Equations for Blood Circulation

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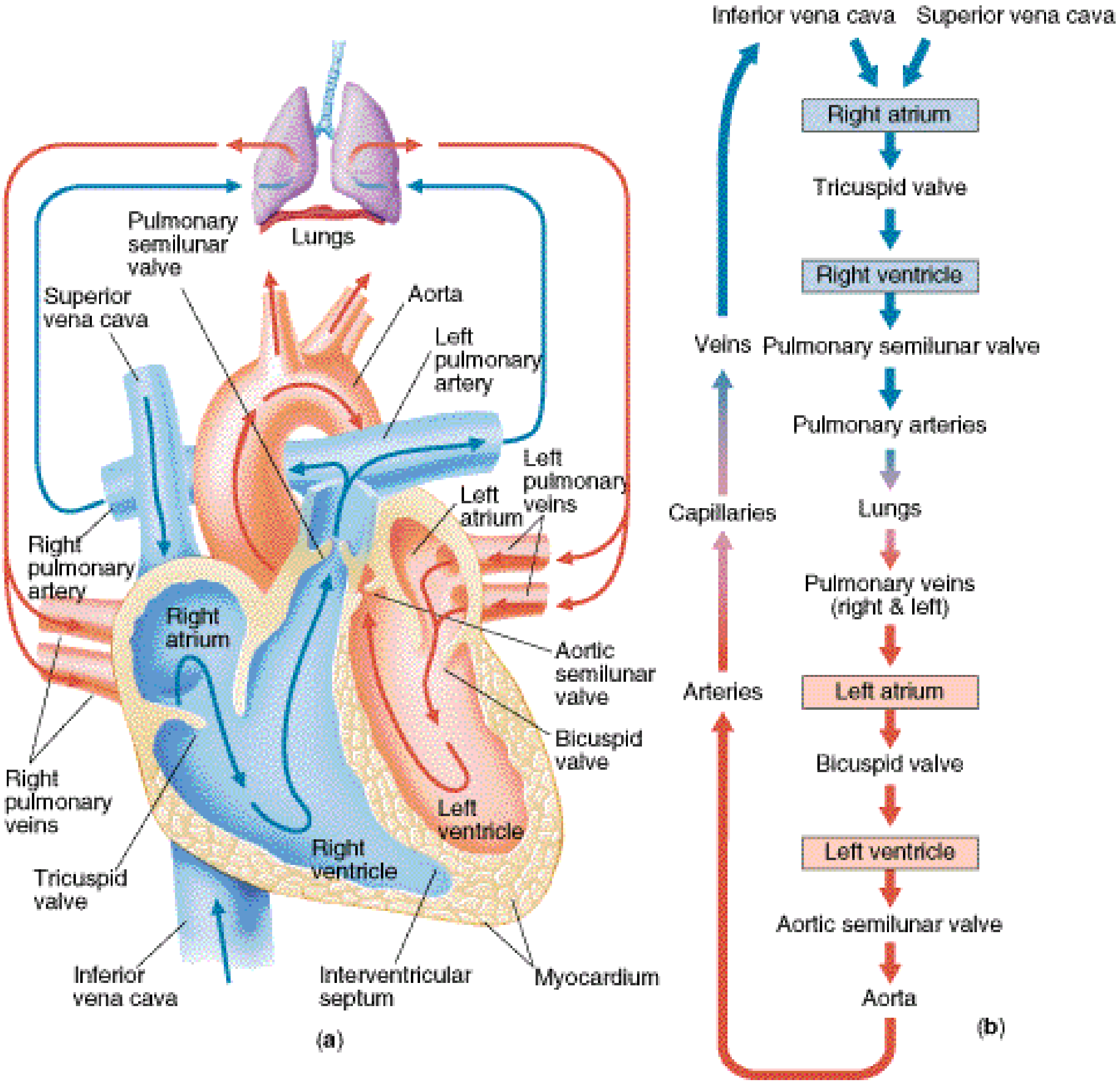
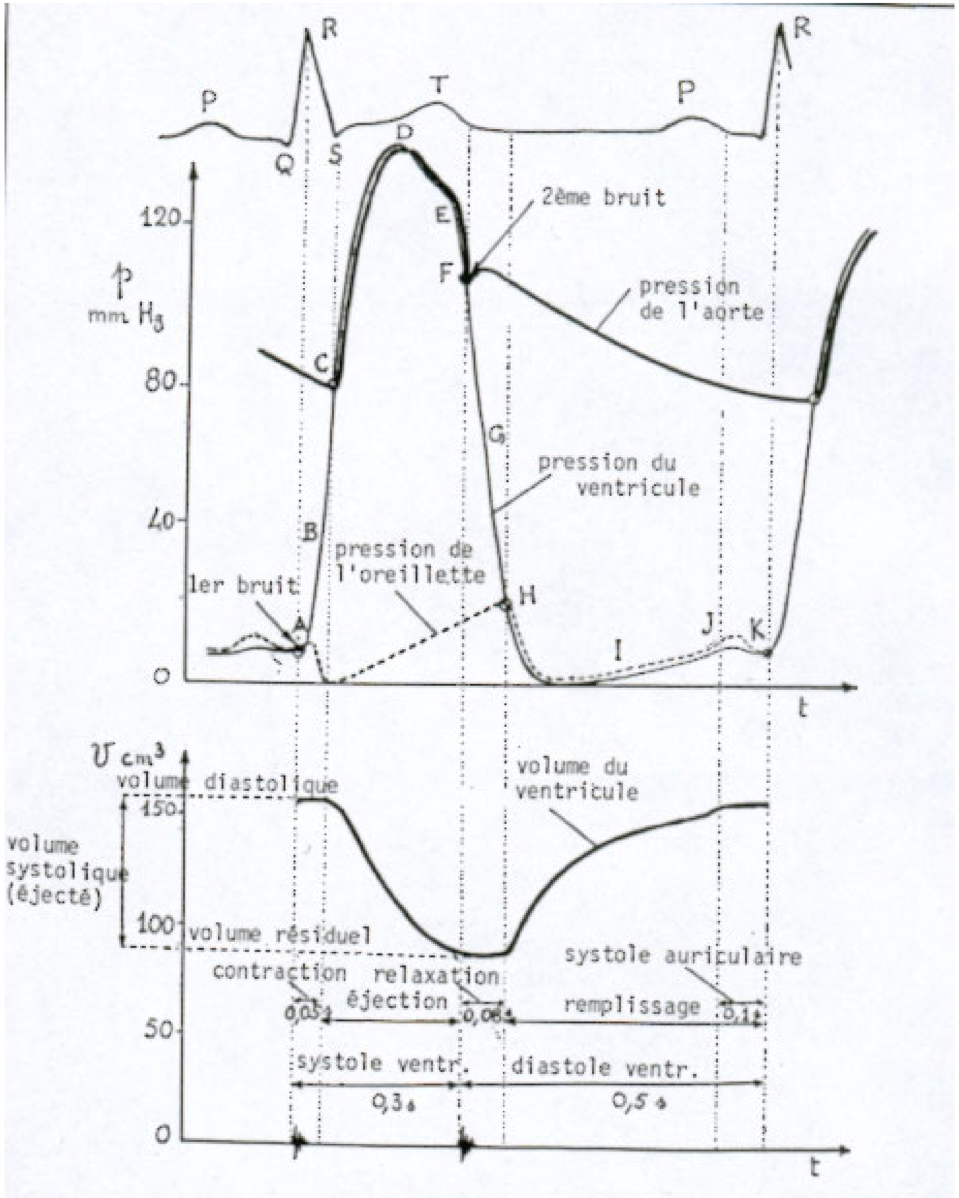
HEART and CIRCULATION



Ref: www-rocq1.
inria.fr/Marc.Thiriet

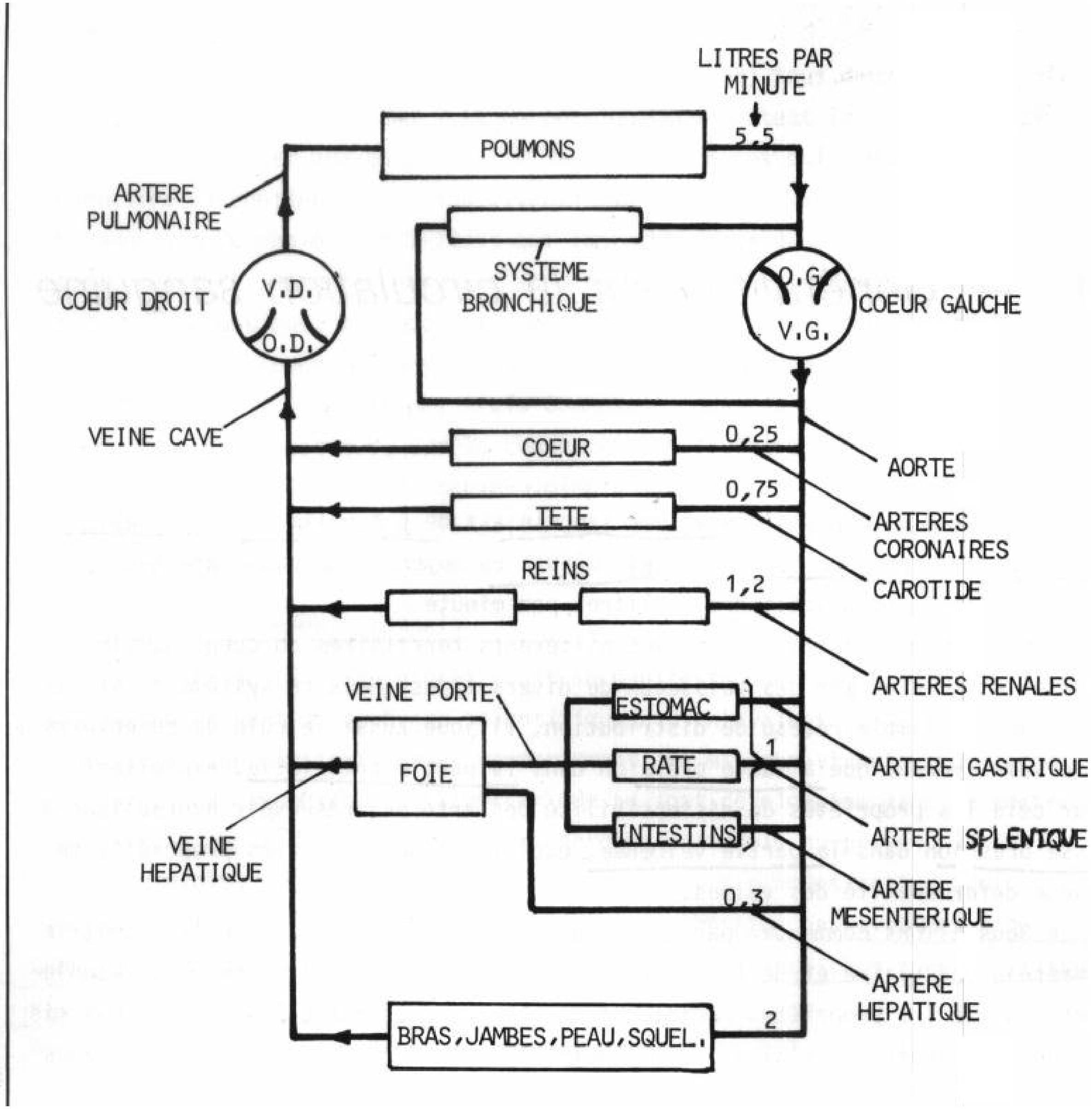
Heart physiological data

(*R. Comolet*
Biomécanique circulatoire,
Ed. Masson, 1984)



ABC: isovolumic contraction ;
CDE: ejection;
FGH: isovolumic relaxation;
HIJK: filling

Circulatory tree



(*R. Comolet Biomécanique circulatoire, Ed. Masson, 1984*)

Arterial and veinous network: physiological data

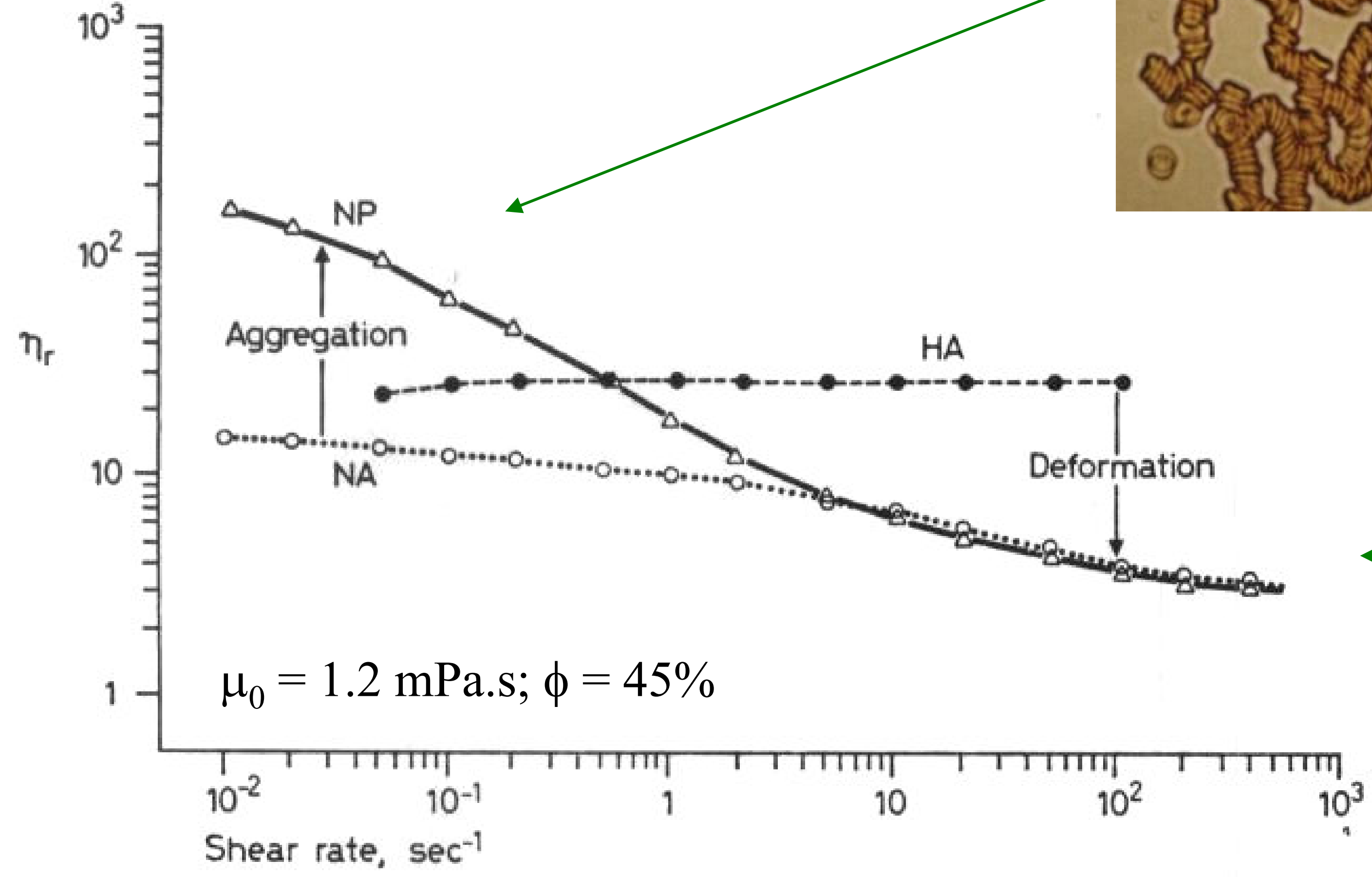
	Diamètre D (cm)	Nombre de vaisseaux	Longueur (cm)	Section totale équivalente (cm^2)	Vitesse U (cm/s)	Nombre de Reynolds
Aorte	2	1	59	3.14	24	1200
	0.5	40	29	8	9.3	116
	0.07	1800	5.8	11	7	12.2
	0.005	1 000 000	0.15	20	3.7	0.46
Micro-capillaires	0.0008	$3\ 10^9$	0.15	1500	0.05	0.001
	0.0075	15 000 000	0.22	670	0.11	0.02
	0.03	76000	2.1	54	1.4	1.05
	1	40	29	32	2.35	58.7
Veine Cave	2.5	1	59	4.8	15.5	968

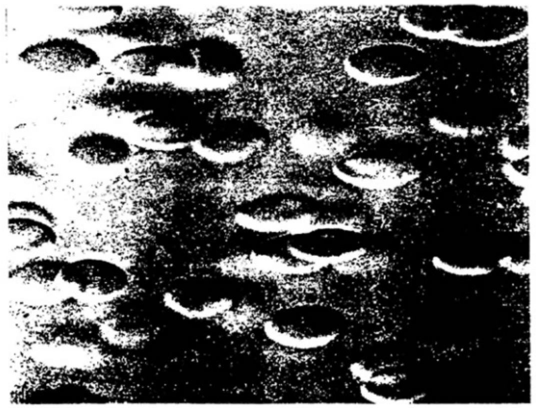
(*R. Comolet Biomécanique circulatoire, Ed. Masson, 1984*)

Reynolds: $\Re_e = \dfrac{\rho\ UD}{\mu}$

Blood viscosity

S. Chien's curve (1970)





$G = 500s^{-1}$; $\mu_0 = 12$mPa.s

NP, NA: Normal RBCs in Plasma or in Albumin solution
HA: Hardened RBCs in Albumin solution

Fischer and Schmid-Schönbein, Blood Cells (3), 1977

Rheology of Fluids: definitions

- Newtonian fluid:
  Its viscosity does not depend on shear rate

- Shear thinning fluid:
  Its viscosity decreases when shear rate increases

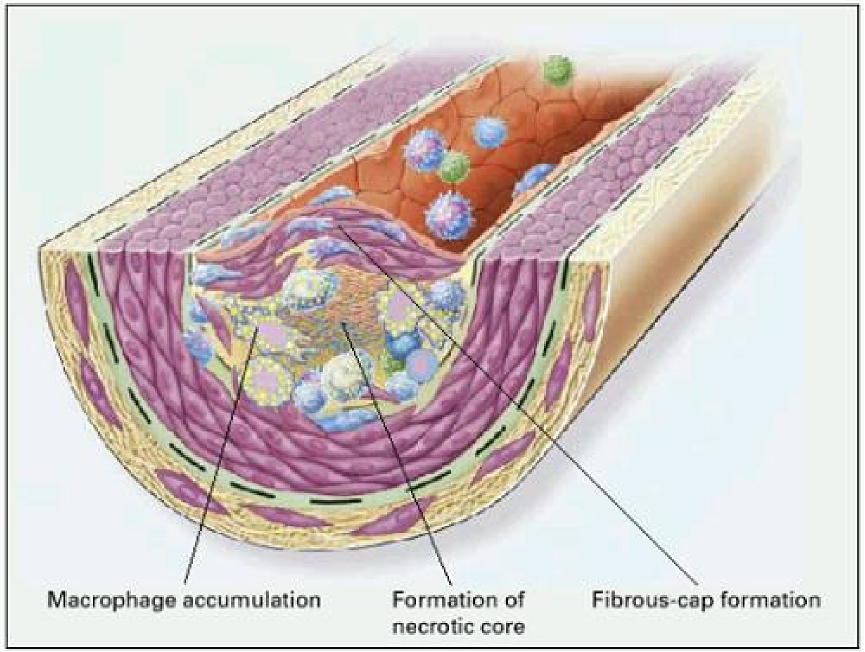
- Yield stress fluid:
  Begins to flow only if the stress is higher than a critical value

- Thixotropic fluid:
  Its viscosity decreases with time, under constant shear rate.
  Restructuration is possible if flow is stopped.

→ Viscosity decreases when temperature increases:
Ex.: water: at 20°C: μ = 1 mPa.s; at 60°C: μ = 0.6 mPa.s

CLINICAL MOTIVATIONS

→ **Atherosclerosis**



R. Ross, N Engl J Med, 1999

→ **Medical device design**

→ **Surgical planning**

→ **Drug transport**

→ **Hyperviscosity, hyperagregability, …**

→ **Coupling with 3D imaging: patient's specific predictions**

→ **Stents, prosthesis**

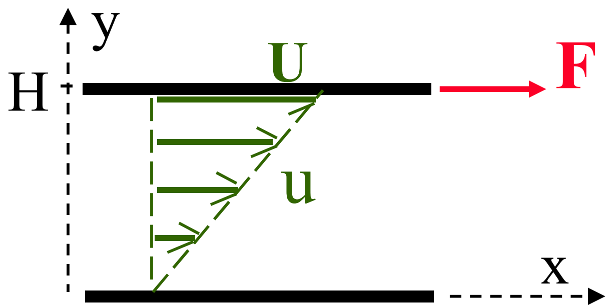
→ **Extracorporeal circulation**

→ **Medical imaging**

Fluid mechanics: basic definitions

Density = ρ (in kg/m^3)

Dynamic viscosity = μ (in Pa.s). (Resistance to flow)

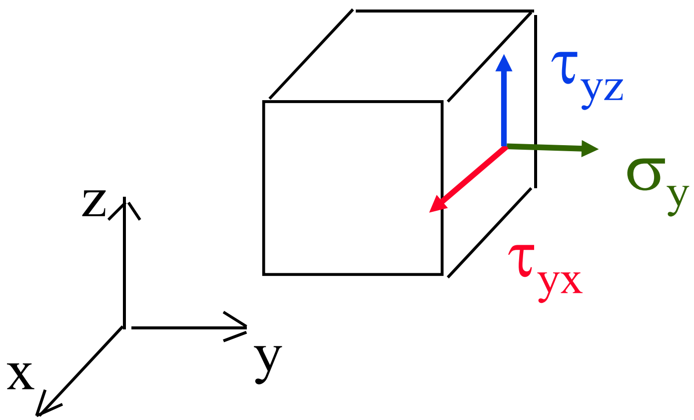


$$\tau = \mu \frac{du}{dy}$$

τ (in Pa): shear stress

du/dy = shear rate (s^{-1})

Stresses on an elementary fluid volume:



$$[\sigma_{ij}] = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Navier-Stokes equations

Momentum equation:

$$\frac{\partial(\rho u_i)}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = \rho f_i + \frac{\partial \sigma_{ik}}{\partial x_k}$$

+

Incompressible newtonian fluid:

$$\sigma_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

↓

Navier-Stokes equations:

$$\begin{cases} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_1 - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_2 - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho f_3 - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \end{cases}$$

+ mass conservation: $\frac{\partial u_i}{\partial x_i} = 0$

Equations in cylindrical coordinates

→ Mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_x)}{\partial x} = 0$$

→ Navier-Stokes equations

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_x\frac{\partial u_r}{\partial x}\right) = -\frac{\partial P}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial x^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right\}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_x\frac{\partial u_\theta}{\partial x}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial x^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right\}$$

$$\rho\left(\frac{\partial u_x}{\partial t} + u_r\frac{\partial u_x}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_x}{\partial \theta} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_x}{\partial \theta^2} + \frac{\partial^2 u_x}{\partial x^2}\right\}$$

→ Shear stresses

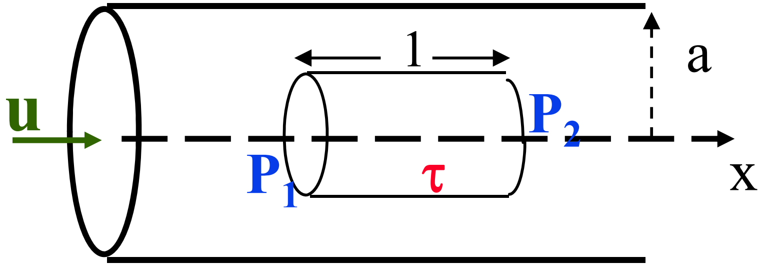
$$\sigma_{rr} = -P + 2\mu\frac{\partial u_r}{\partial r} \qquad \tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

$$\sigma_{\theta\theta} = -P + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \qquad \tau_{\theta x} = \tau_{x\theta} = \mu\left[\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial \theta}\right]$$

$$\sigma_{xx} = -P + 2\mu\frac{\partial u_x}{\partial x} \qquad \tau_{xr} = \tau_{rx} = \mu\left(\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r}\right)$$

Laminar steady flow in a rigid tube – Poiseuille law

Forces Equilibrium: $\tau\, 2\pi r\, 1 = -\pi r^2 \frac{dP}{dx}$



Newtonian fluid : $\tau = -\mu \frac{du}{dr}$

Velocity profile : $u = -\frac{1}{4\mu}\left(a^2 - r^2\right)\frac{dP}{dx}$

Poiseuille law:

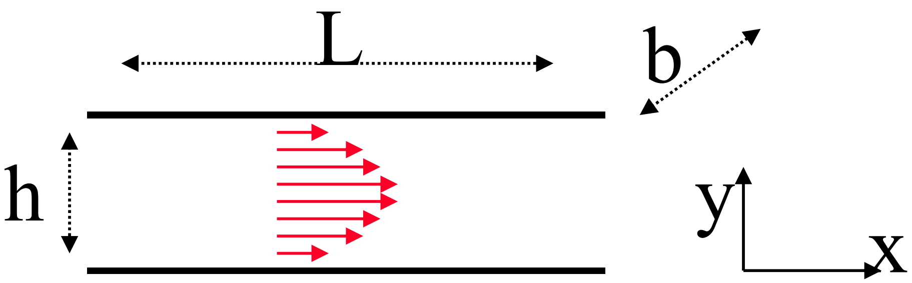
Flow rate Q: $Q = \int_0^a u(r)\, 2\pi r\, dr \longrightarrow$

$$\Delta P = \frac{8\mu L}{\pi a^4} Q$$

Max velocity: $\mathbf{U_{max} = 2\, U_{moy}}$

Wall shear stress: $\tau_w = \frac{4\mu}{\pi a^3} Q$

Newtonian viscous incompressible fluid, Laminar flow between 2 « infinite » parallel plates: Plane Poiseuille law



h<<b et L

Velocity profile :

$$u(y) = -6U_m\left(\frac{y^2}{h^2} - \frac{y}{h}\right)$$

Flow rate: $Q = \int_0^h u(y)b\,dy$ with $U_m = \frac{Q}{bh}$

Wall shear stress: $\tau_w = \frac{6\mu}{b\,h^2}Q$

Plane Poiseuille law: $\Delta P = \frac{12\,\mu L}{b\,h^3}Q$

Sinusoïdal flow in a rigid tube – Womersley flow

Fully developed unidirectional and axisymmetric flow: $\frac{\partial u_x}{\partial x} = 0;\ u_r = 0; u_\theta = 0$

Equation of motion: $\frac{\partial u_x}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_x}{\partial r})$

Boundary conditions: $u_x(a,t) = 0\ and\ \frac{\partial u_x}{\partial r} = 0\ at\ r = 0$

Pressure gradient: $\frac{\partial P}{\partial x} = \frac{\partial P_1}{\partial x}e^{i\omega t}$

Non-dimensional form of the equation: $u' = \frac{u_x}{U};\quad r' = \frac{r}{a};\quad P' = \frac{P}{\rho U^2};\quad t' = \omega t$

$$\alpha^2 \frac{\partial u'}{\partial t'} = -\Re e\frac{\partial P'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}(r'\frac{\partial u'}{\partial r'})$$

Womersley parameter: $\alpha = a\sqrt{\frac{\rho\omega}{\mu}}$

→ If α small, equilibrium of viscous forces and the driving pressure gradient

A quasi-static Poiseuille profile is found $u_x(r,t) = -\frac{1}{4\mu}(a^2 - r^2)\frac{\partial P_1}{\partial x}e^{i\omega t}$

Womersley flow (2)

→ If α large, equilibrium of inertia forces and the driving pressure gradient

A plug flow is found $u_x(r,t) = \frac{i}{\rho\omega}\frac{\partial P_1}{\partial x}e^{i\omega t}$ (π/2 out of phase with the pressure gradient)

→ For intermediate α values, all terms remain in the equation

$$u_x(r,t) = \frac{i}{\rho\omega}\left[1 - \frac{J_0(i^{3/2}\alpha\frac{r}{a})}{J_0(i^{3/2}\alpha)}\right]\frac{\partial P_1}{\partial x}e^{i\omega t}$$

(J_0 = 0-th order Bessel function of the first kind)

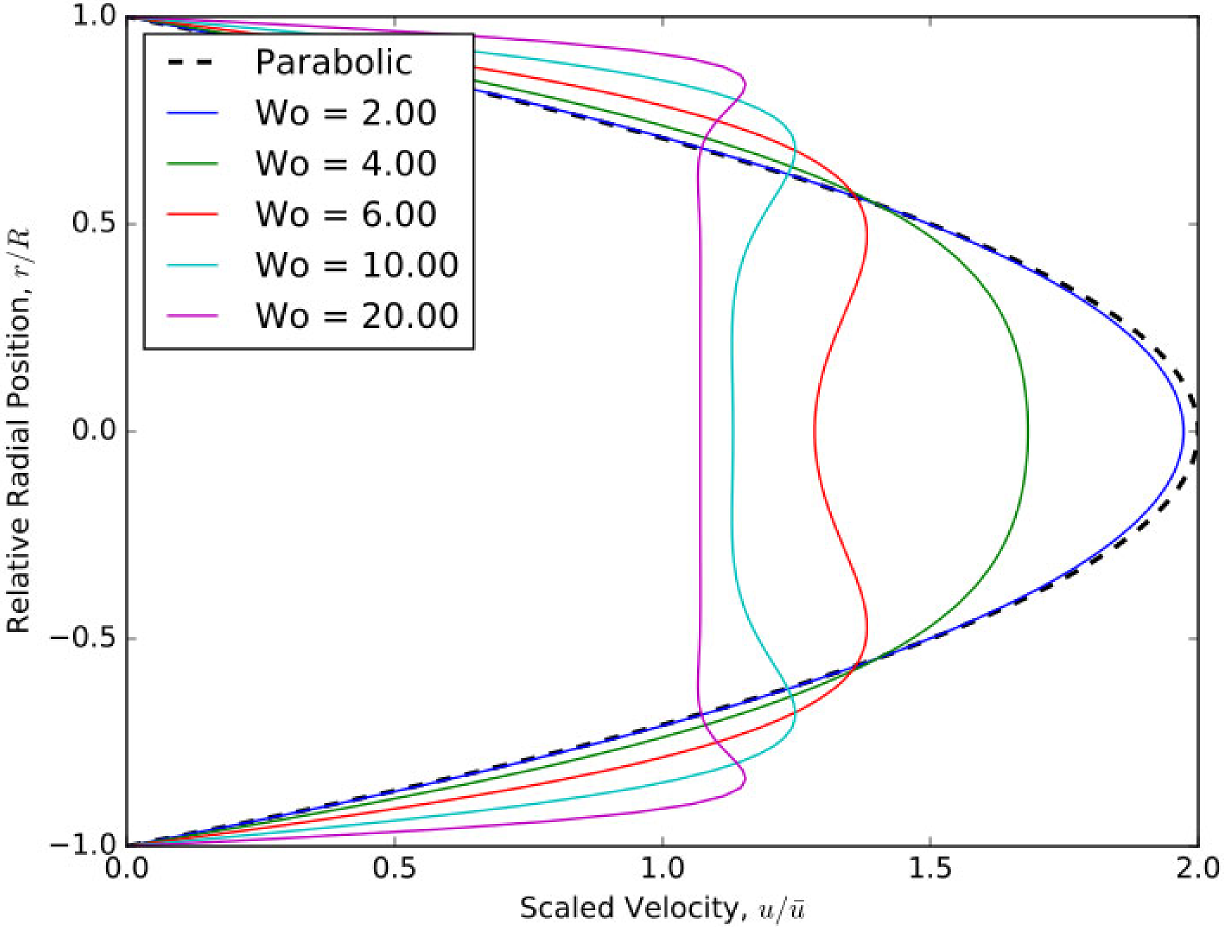
Viscous forces dominant near the wall and flat profile in the core

Corresponding flow rate:

$$q = \frac{i\pi a^2}{\rho\omega}\left[1 - F_{10}(\alpha)\right]\frac{\partial P_1}{\partial x}e^{i\omega t} \quad \text{with} \quad F_{10}(\alpha) = \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha J_0(i^{3/2}\alpha)}$$

(J_1 = 1st order Bessel function of the first kind)

Womersley flow (Illustration)



From Rudolf Helmuth, 2017

Pulsed flow in a distensible tube

- Problem to solve: Fluid motion

+

Wall displacement (ξ_r and ξ_x) [depend on the mechanical properties of the wall]

+

Boundary conditions: At r = 0: $u_r = 0 \;\; and \;\; \frac{\partial u_x}{\partial r} = 0$

At the fluid-solid interface (r = a): continuity of velocities and stresses

$$u_r = \frac{\partial \xi_r}{\partial t} \;\; and \;\; u_x = \frac{\partial \xi_x}{\partial t}$$

- Approximation: case where the wave speed is large (U/c<<1) and the wave length is long (a/λ<<1): locally, the momentum equation as found for the rigid tube may be used

Using the dimensionless quantities: $r' = \frac{r}{a}; \quad x' = \frac{x}{\lambda}; \quad u' = \frac{u_x}{U}; \quad u'_r = \frac{u_r}{U}\frac{\lambda}{a}; \quad t' = \frac{c}{\lambda}t; \quad P' = \frac{P}{\rho c U}$

Axial projection of N.S. equa: $$\frac{\partial u'}{\partial t'} + \underbrace{\frac{U}{c}(u'_r \frac{\partial u'_x}{\partial r'} + u'_x \frac{\partial u'_x}{\partial x'})}_{\text{neglected}} = -\frac{\partial P'}{\partial x'} + \frac{2\pi}{\alpha^2}[\frac{1}{r'}\frac{\partial}{\partial r'}(r'\frac{\partial u'_x}{\partial r'}) + \underbrace{\frac{a^2}{\lambda^2}\frac{\partial^2 u'_x}{\partial x'^2}}_{\text{neglected}}]$$

Bernoulli's equation

- Conservation de l'énergie mécanique totale le long d'une ligne de courant pour un fluide parfait, incompressible, en écoulement stationnaire

$$P + \rho g z + \frac{1}{2}\rho V^2 = Cste$$

P : pression statique ; ρgz : pression due à la pesanteur ;
$\frac{1}{2} \rho V^2$: pression dynamique (V: vitesse de l'écoulement)

Ex: vérifier que tous les termes de l'équation sont homogènes à des Pascals, et que les Pascals représentent une énergie par unité de volume

- Si le fluide n'est pas parfait (fluide visqueux), il se produit une perte d'énergie au long de l'écoulement, due aux frottements. On écrit alors l'équation de Bernoulli généralisée.
  Pour un écoulement stationnaire d'un fluide incompressible, d'un point A vers un point B le long d'une ligne de courant:

$$P_A + \rho g z_A + \frac{1}{2}\rho V_A{}^2 = P_B + \rho g z_B + \frac{1}{2}\rho V_B{}^2 + \Delta P_f$$

Les pertes de charge par frottement, ΔP_f, sont proportionnelles à la pression dynamique:

$$\Delta P_f = \zeta\,(\tfrac{1}{2}\rho V^2)$$

Le coefficient ζ dépend de la nature et de la géométrie de l'écoulement

Bernoulli's equation: pressure drops

- Ecoulement laminaire dans une conduite cylindrique de rayon a et longueur l :

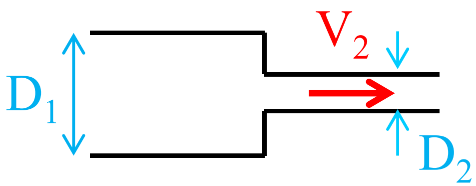
$$\zeta = \frac{64}{\mathcal{R}_e}\frac{l}{2a}$$

où R_e est le nombre de Reynolds, donné par : $\mathcal{R}_e = \frac{\rho V 2a}{\mu}$

Ex1: vérifier que le nombre de Reynolds est un nombre sans dimension.

Ex2: montrer que, dans ce cas, on obtient la relation: $\Delta P_f = \frac{8\mu l}{\pi a^4} Q$

- Rétrécissement brusque de la canalisation:



$$\zeta = \frac{1}{2}\left(1 - \frac{D_2^{\,2}}{D_1^{\,2}}\right) \qquad \Delta P_f = \zeta \, ½ \, \rho \, V_2^{\,2}$$

- Elargissement brusque de la canalisation:

$$\zeta = \left(1 - \frac{D_1^{\,2}}{D_2^{\,2}}\right)^2 \qquad \Delta P_f = \zeta \, ½ \, \rho \, V_1^{\,2}$$

LUMPED PARAMETER MODELS

Windkessel model – O. Franck (1899)

Cardiac valve + elastic chamber (arterial compliance C) + peripheral resistance R

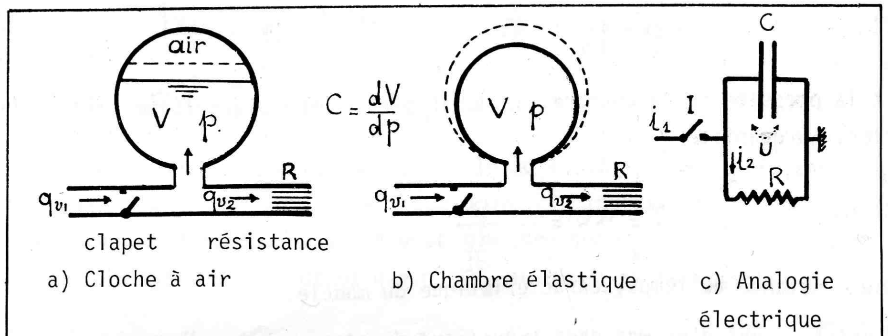


Fig. VI-3. - Modèles "Windkessel"

(From Comolet, 1984, Masson)

Electric analogy

- Pressure drop ΔP / voltage U
- Flow rate Q / current I
- Resistance: $U = R\, I$
- Compliance: $U = \frac{1}{C}\int I\, dt$

- Cardiac valve closed (diastole):

Blood stored in the chamber flows through the resistance

$$\frac{dV}{dt} + q_{v2} = 0\ ;\quad q_{v1} = 0;\quad P = Rq_{v2}$$

→ Differential equation: $C\frac{dP}{dt} + \frac{P}{R} = 0$

→ Solution: $P = P_0\, e^{-t/RC}$

P_0 = pressure at the beginning of diastole (= end of systole);

P_1 = pressure at the end of diastole ($t = t_1$)

→ $P_1 = P_0\, e^{-t_1/RC}$

2-element Windkessel model (2)

- Cardiac valve open (systole):

$$q_{v1} = \frac{dV}{dt} + q_{v2}; \quad P = Rq_{v2}$$

- Differential equation:

$$RC\frac{dP}{dt} + P = Rq_{v1} = RA$$

- Solution:

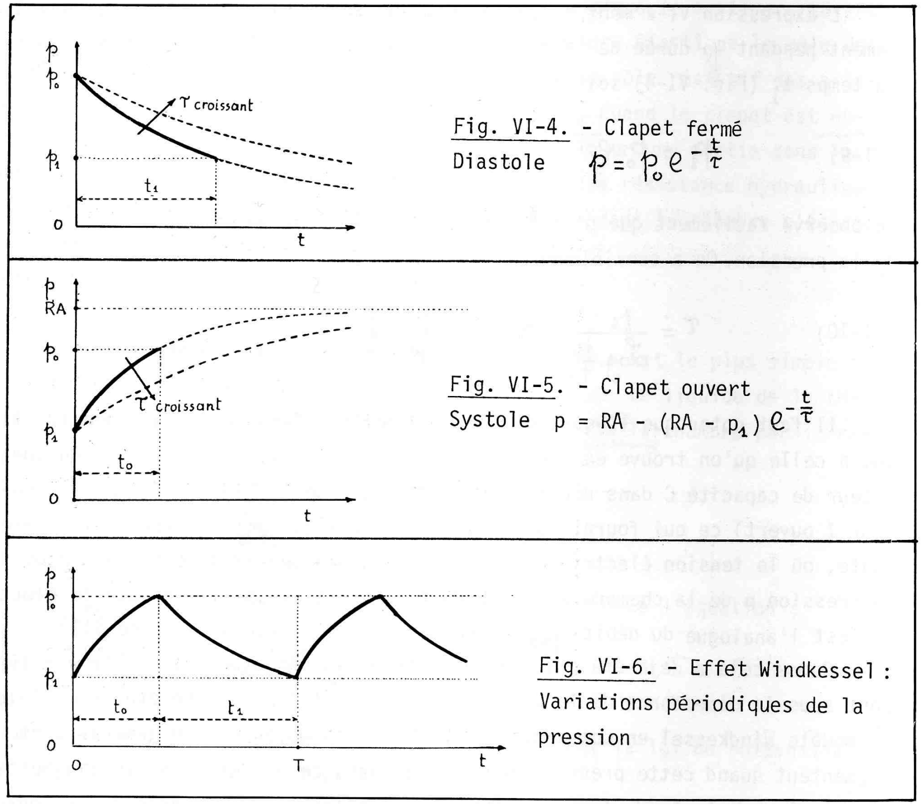
$$P = RA + (P_1 - RA)e^{-t/RC}$$

P_1 = pressure at the beginning of systole
P_0 = pressure at the end of systole ($t = t_0$)

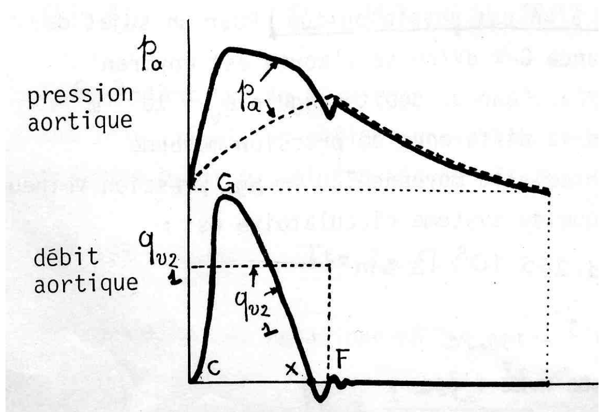
$$P_0 = RA + (P_1 - RA)e^{-t_0/RC}$$

→ P varies between P_0 and P_1:

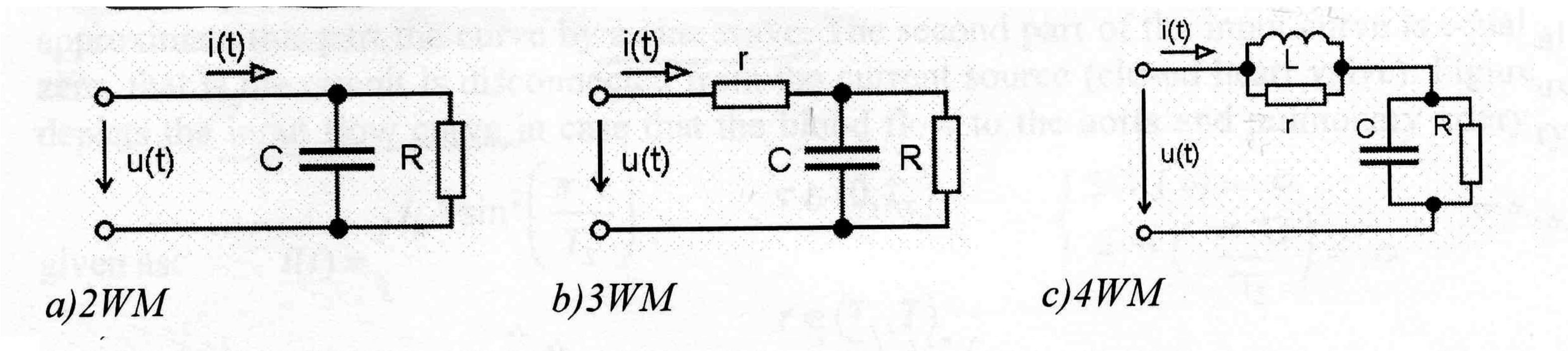
$$P_0 = RA\frac{[1 - e^{-t_0/RC}]}{[1 - e^{-T/RC}]} \; and \; P_1 = RA\frac{[1 - e^{t_0/RC}]}{[1 - e^{T/RC}]}$$



(From Comolet, 1984, Masson)



Extension of Windkessel models



$$U = L\frac{dI}{dt}$$

3 elt-WM: R of aorta is added; 4 elt-WM: an inductor is added, representing inertia of blood

Differential equations:

3 elt-WM: $$(R+r)i(t) + RrC\frac{di(t)}{dt} = u(t) + RC\frac{du(t)}{dt}$$

4 elt-WM: $$CLrR\frac{d^2i(t)}{dt^2} + L(r+R)\frac{di(t)}{dt} + rRi(t) = CLR\frac{d^2u(t)}{dt^2} + (L+CrR)\frac{du(t)}{dt} + ru(t)$$

(Olufsen and Nadim, Math. Biosciences and Engineering, 2004)

→ ## Lumped parameter models:

- No spatial dependency; time variable only
- Consider elementary components of the circulatory tree as « compartments »
- Flow rate and pressure are averaged in space over the whole compartment
- Wave propagation and reflection cannot be incorporated

Parameter determination

The R, L, C parameters depend on the blood and vessel wall mechanical properties

→ For a vessel of length l, radius a, wall thickness h, wall Young modulus E:

- Pietrabissa et al. (*Med. Engin. Physics, 1996*) $R = \frac{8\mu l}{\pi a^4}$ $L = \frac{\rho l}{\pi a^2}$ $C = \frac{2\pi a^3 l}{Eh}$

- Quarteroni et al. (*Computing and Visualization in Science, 2001*) $C = \frac{9\pi a^3 l}{8Eh}$

  (σ = Poisson ratio = ½, if vessel wall considered as incompressible)

  http://mox.polimi.it/it/progretti/pubblicazioni

- Comolet (*Biomécanique circulatoire, 1984*) $C = \frac{3}{2}\frac{\pi a^3 l}{Eh}\frac{(1+\frac{h}{a})^2}{(1+\frac{h}{2a})}$

Introduction to 1D models

- 1D non linear systems which model the blood pulse propagation in compliant arteries

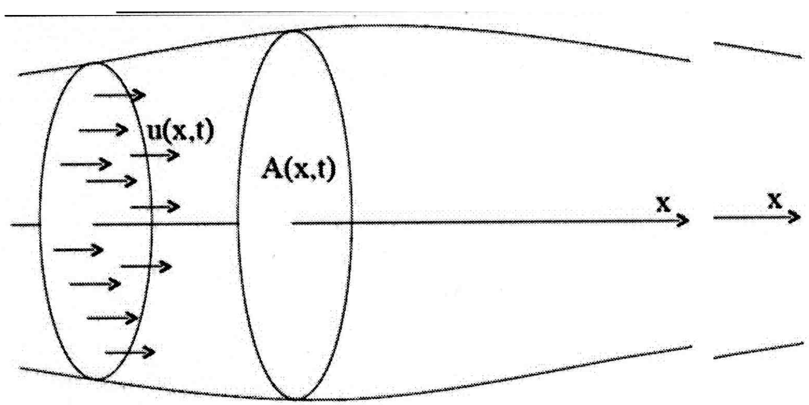
- Obtained by averaging the N. Stokes equ. on each section of an arterial vessel and using simplified models for the vessel compliance

Simplifying assumptions

- The dominent component of blood flow velocity is oriented along the vessel axis and pressure can be assumed constant (or averaged) over the cross-section of the vessel

- Fixed cylinder axis and axial symmetry

- Radial wall displacements ($a-a_0$)



$$q = 2\pi \int_0^a u_x(r,x,t) r dr \qquad \frac{\partial A}{\partial t} = 2\pi a \frac{\partial a}{\partial t}$$

1D models equations

→ 2 ways of deriving 1D models equations

- Integration of the axial projection of N. Stokes equ, with asymptotic analysis by assuming that a_0/l is small *(Olufsen et al., Annals of Biomedical Engineering, 2000)*

- Basic conservation laws written in integral form *(Formaggia, Veneziani, and coll.)*

→ Coupled system of differential equations (unknowns = A, q, P)

- Continuity equation $$\frac{\partial q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

- Momentum equation → Olufsen's formulation $$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(2\pi \int_0^a u_x^2 r dr) + \frac{A}{\rho}\frac{\partial P}{\partial x} = 2\pi \frac{\mu}{\rho}[r \frac{\partial u_x}{\partial r}]_a$$

↓

Mox team formulation $$\frac{\partial q}{\partial t} + \alpha \frac{\partial}{\partial x}(\frac{q^2}{A}) + \frac{A}{\rho}\frac{\partial P}{\partial x} + K_r \frac{q}{A} = 0$$

with $$\begin{cases} u_x(r,x,t) = \overline{u}(x,t)s(y) \\ y = \dfrac{r}{a(x,t)} \end{cases}$$ and $$\begin{cases} \alpha = \dfrac{1}{A\overline{u}^2}\displaystyle\int_A u_x^2 dA \\ K_r = -2\pi \dfrac{\mu}{\rho}[s'(y)]_{y=1} \end{cases}$$

s = velocity profile law
α = momentum correction coeff
K_r = resistance parameter

1D models equations (2)

→ Choice of a velocity profile

Olufsen's formulation

$$u_x = \begin{cases} \bar{u}_x \;\; for\; r \le a-\delta \\ \bar{u}_x \dfrac{(a-r)}{\delta} \;\; for\; a-\delta < r < a \end{cases}$$

(Flat velocity profile except in a thin boundary layer of width δ<<a)

Mox team formulation

$$s(y) = \frac{(\gamma+2)}{\gamma}(1-y^{\gamma})$$

- When γ = 2, Poiseuille profile
- The higher γ, the flater the profile

- State equation

Olufsen's formulation

$$P(x,t) - P_0 = \frac{4}{3}\frac{Eh}{a_0(x)}\left[1-\sqrt{\frac{A_0(x)}{A(x,t)}}\right]$$

$a_0(x)$: shape of the vessel when transmural pressure is zero

Mox team formulation

$$P(x,t) - P_0 = \beta(\sqrt{A} - \sqrt{A_0}) \;\; with \;\; \beta = \frac{\sqrt{\pi}Eh}{(1-\sigma^2)A_0}$$

(σ = Poisson coefficient)

Equivalence of the 2 formulations

Olufsen's formulation

$$q = A\overline{u}_x(1 - \frac{\delta}{a} + O(\delta^2))$$

$$2\pi\frac{\mu}{\rho}[r\frac{\partial u_x}{\partial r}]_{r=a} = -2\pi\frac{\mu}{\rho}\frac{a}{\delta}(\frac{q}{A})(1 + O(\delta))$$

$$2\pi\int_0^a u_x^2 r dr = \frac{q^2}{A}(1 + \frac{2}{3}\frac{\delta}{a} + O(\delta^2))$$

↓

Momentum equation

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(\frac{q^2}{A}) + \frac{A}{\rho}\frac{\partial P}{\partial x} + 2\pi\frac{\mu}{\rho}\frac{a}{\delta}(\frac{q}{A}) = 0$$

Suggested value of a/δ:
about 10, for large arteries

Mox team formulation

$$K_r = 2\pi\frac{\mu}{\rho}(\gamma + 2)$$

$$\alpha = \frac{\gamma + 2}{\gamma + 1}$$

↓

Momentum equation

$$\frac{\partial q}{\partial t} + (\frac{\gamma + 2}{\gamma + 1})\frac{\partial}{\partial x}(\frac{q^2}{A}) + \frac{A}{\rho}\frac{\partial P}{\partial x} + 2\pi\frac{\mu}{\rho}(\gamma + 2)(\frac{q}{A}) = 0$$

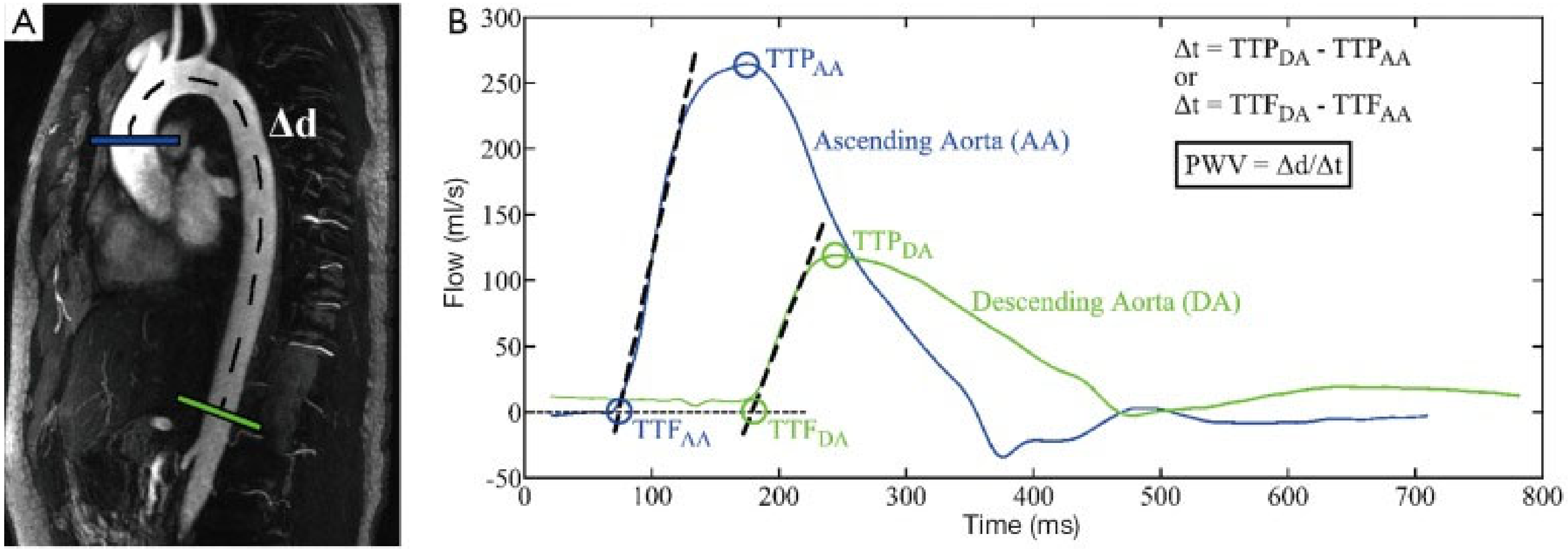
Suggested value of γ: γ = 9

Exemple de mesure non invasive

Le module d'Young de la paroi aortique

⟶ Mesure de la célérité de l'onde de pouls (déf. radiale) par IRM

Temps nécessaire à l'onde pour aller d'une section de l'artère à une autre: PWV = Δd /Δt



Wentland et al. (2014) Cardio Vasc Diagn Ther

⟶ Formule de Moens-Korteweg

$$PWV = \sqrt{\frac{E\,h}{2\rho R}}$$

E = module Young paroi artère (Pa)
(hyp: isotrope, mince, élastique)

h, R = épaisseur et rayon du vaisseau (m)

ρ = masse volumique du sang (kg/m^3)

Le module d'Young de la paroi aortique (suite)

- Ordre de grandeur (non pathologique): R = 1 cm, h = 2 mm, ρ = 1050 kg/m^3, PWV environ 9 m/s, E environ 10^6 Pa.

- Altérations par vieillissement, maladies inflammatoires, athérosclérose, diabète, calcification, … : risque cardio-vasculaire

- Avantage de la méthode: tient compte de l'influence des tissus environnants (muscles, graisse, os, …) sur la paroi artérielle

- Si champ B très élevé, B peut avoir une influence sur PWV

A. Drochon (2016) « Sinusoïdal flow of blood in a cylindrical deformable vessel exposed to an external magnetic field » Eur. Phys. J. App. Phys. 73:31101

Starling law

→ Represents the fluid flow across the endothelial cells layer

$$J = AL_p[(P_c - P_i) - \sigma(\pi_p - \pi_i)]$$

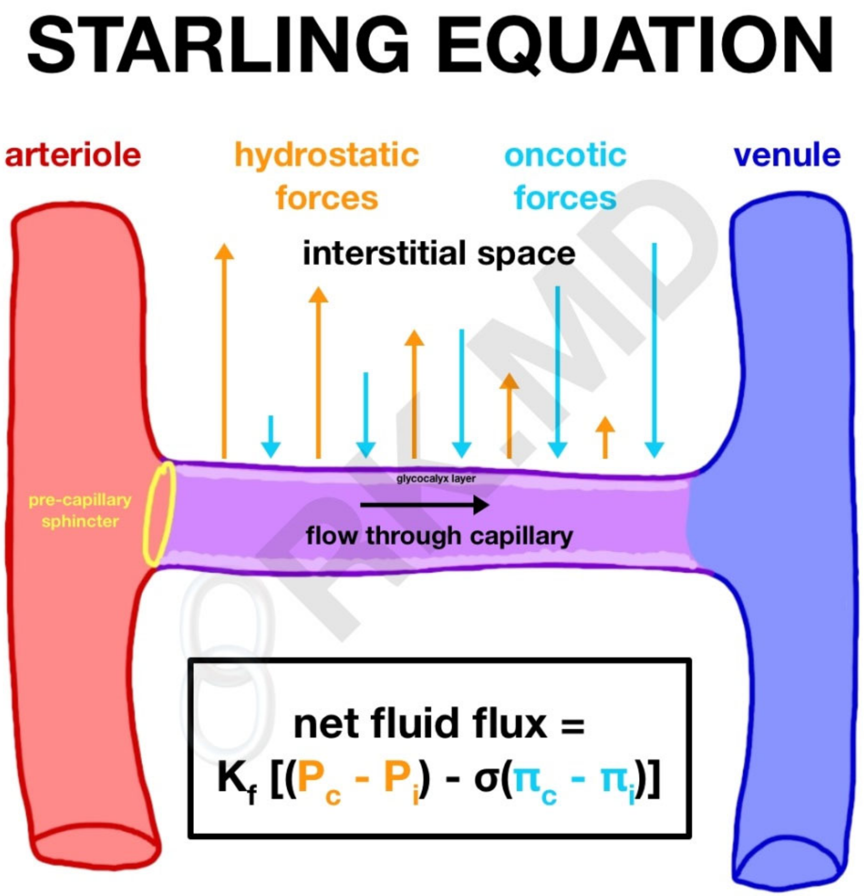
J = rate of fluid movement (m^3 /s)
L_p = Hydraulic conductivity of the endothelium (m /Pa.s)
A = area of the capillary wall available for filtration (m^2)
P_c = Capillary hydrostatic pressure (Pa)
P_i = Interstitial fluid (tissue) hydrostatic pressure (Pa)
σ = reflection coefficient (no unit)
Π_p = plasma protein oncotic pressure (Pa)
Π_i = interstitial fluid oncotic pressure (Pa)

- Albumin is responsible for about 70% of the oncotic pressure

Plasma protein conc. (g/l)	40	60	80
Oncotic pressure (mmHg)	10	20	30

- The reflection coefficient σ for protein permeability is specific for each membrane and protein
  σ = 0 : membrane is maximally permeable to the protein;
  σ = 1: membrane is totally impermeable

Starling law (2)



https://rk.md/2018/starling-forces-hydrostatic-and-oncotic-pressures/

- In the postarteriolar capillary segments, the hydrostatic pressure is greater than the oncotic pressure, favouring the movement of water into the interstitial fluid; in the postcapillary venules, oncotic pressure is greater than hydrostatic pressure, favouring the movement of water out of the interstitial fluid to the venules

- Oncotic pressure = colloïdo-osmotic pressure = osmotic pressure that draws water towards the proteins

Fick's diffusion law

- **Fick's first law: diffusive flux :** the movement of particles from high to low concentrations is proportional to the particle's concentration gradient

$$\vec{J} = -D\,\overrightarrow{grad}\,\emptyset$$

J is the diffusion flux vector (amount of substance / m^2.s)
D is the diffusion coefficient (m^2/s)

grad ϕ is the concentration gradient (amount of substance / m^3.m)

- **Fick's second law:** predicts how diffusion causes the concentration to change with respect to time

$$\frac{\partial \emptyset}{\partial t} = div\left(D\,\overrightarrow{grad}\,\emptyset\right)$$

If D does not depend on space coordinates:

$$\frac{\partial \emptyset}{\partial t} = D\,\Delta\emptyset$$

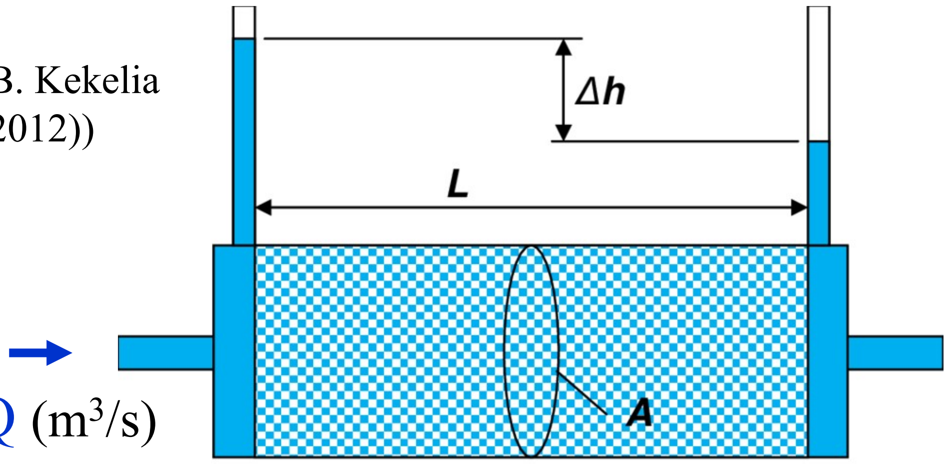
- **Diffusion coefficient for a molecule suspended in a viscous medium:** (Stokes-Einstein equation)

$$D = \frac{k_B\,T}{6\pi\mu a}$$

k_B is Boltzmann constant (= 1.38 10^{-23} m^2.kg / s^2.K)
T is the absolute temperature (K)
μ = dynamic viscosity of the medium (Pa.s)
a = equivalent radius of the molecule (m)

Darcy's law: porous medium

(B. Kekelia (2012))



Q (m³/s)

$$Q = \frac{kA}{\mu L}\Delta P$$

$\Delta P = \rho\, g\, \Delta h$ (Pa)
A = transverse section (m^2)
μ = fluid viscosity (Pa.s)
k = permeability (m^2)

- **Medium porosity: ϕ**

$$\Phi = \frac{volume\ of\ void\ space}{total\ volume}$$

- **Flow velocity: u** (m/s) $u = \frac{1}{\Phi}\frac{Q}{A}$

$q = \frac{Q}{A}$ is not the velocity at which the fluid is travelling through the pores

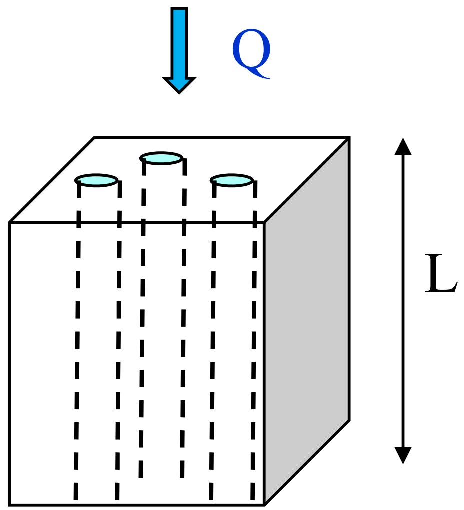
- **Hydraulic conductivity: K** (m/s)
(Flow of water)

$$K = k\frac{\rho g}{\mu}$$

- A medium may be extremely porous, but if the pores are not connected, it will have no permeability. The voids can have different shapes and connectivity, which affects how easily a fluid can move through the pore space. The permeability is a measure of the ease with which liquids and gases can pass through a medium

Darcy's law: example

A parallel bundle of n vertical tubes



Transverse area = A (m^2)
Radius of each pore = a (m)

$$\Phi = porosity = \frac{n\,\pi a^2\,L}{A\,L} = \frac{n\,\pi a^2}{A}$$

Poiseuille law in each pore:

$$\Delta P = \frac{8\mu\,L}{\pi a^4}\left(\frac{Q}{n}\right)$$

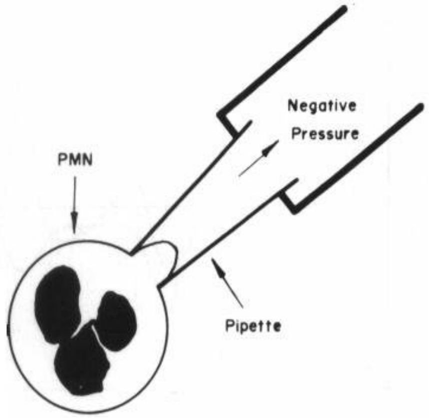
$$Q = \frac{n\,\Delta P\,\pi a^4}{8\mu\,L} = \frac{A\Phi\,\Delta P\,a^2}{8\mu\,L}$$

$$Q = \frac{\Phi a^2}{8}\,\frac{A\,\Delta P}{\mu\,L} = k\,\frac{A\,\Delta P}{\mu\,L}$$

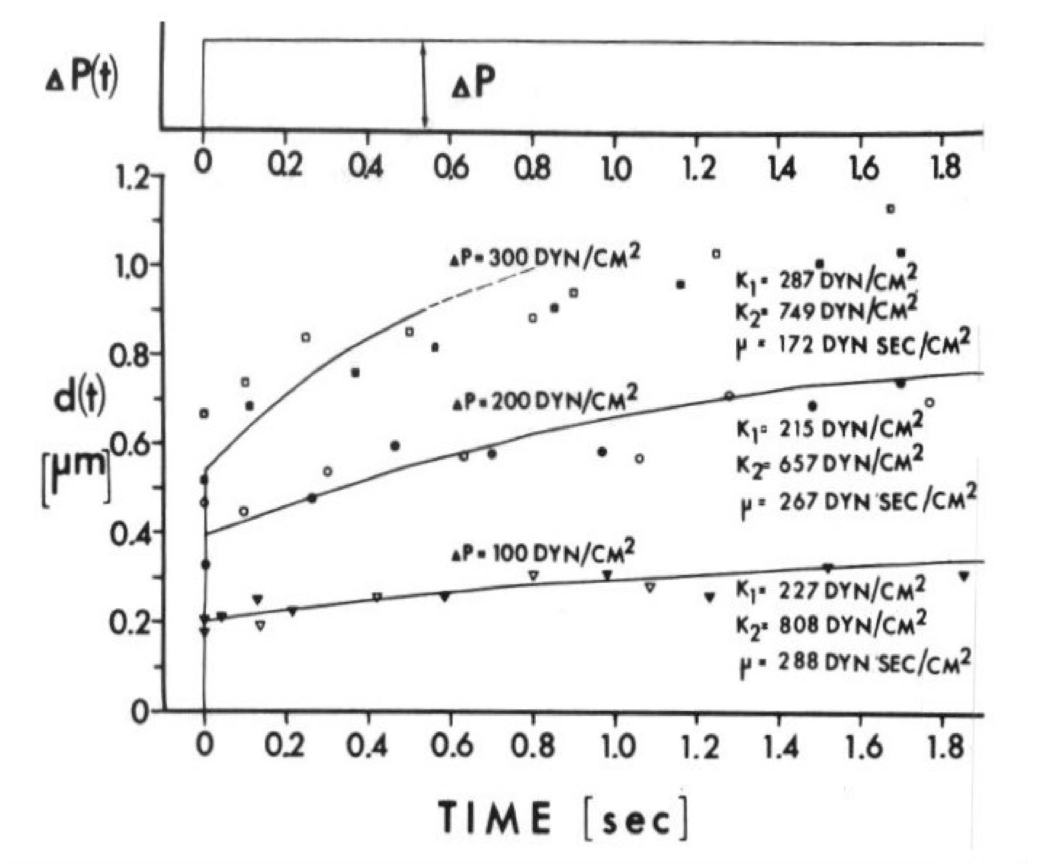
→ Relation between porosity and permeability

$$k = \frac{\Phi a^2}{8}$$

Visco-elasticity:
example with leucocyte micropipette aspiration

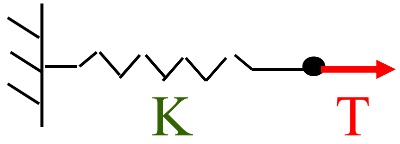


From White Cell Mechanics,
A.R. Liss Ed., 1984

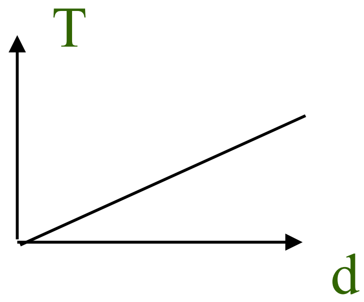


Visco-elasticity: basic tools

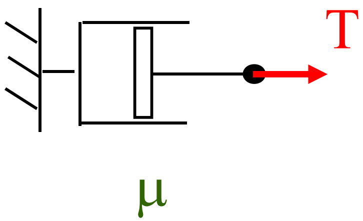
Elastic solid:



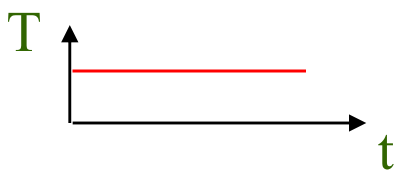
T = Kd

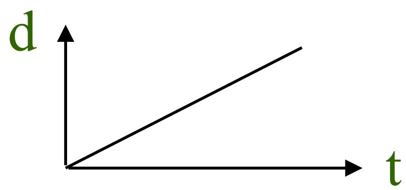


Viscous dashpot:

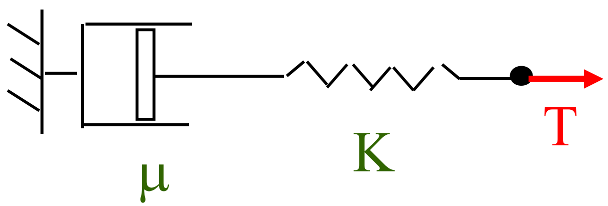


$$T = \mu \dot{d}$$





Maxwell element:



$$T = \mu \dot{d}_1 = K d_2$$
$$d = d_1 + d_2$$

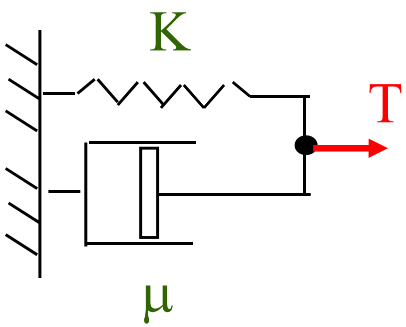
$$\longrightarrow \quad \frac{\dot{T}}{K} + \frac{T}{\mu} = \dot{d} \quad \longrightarrow$$

Relaxation function:

$$T = K d_0 \exp(-\frac{K}{\mu} t)$$

Visco-elasticity: basic tools (2)

Voigt element:



$$T = \mu \dot{d}_1 + K d_2$$

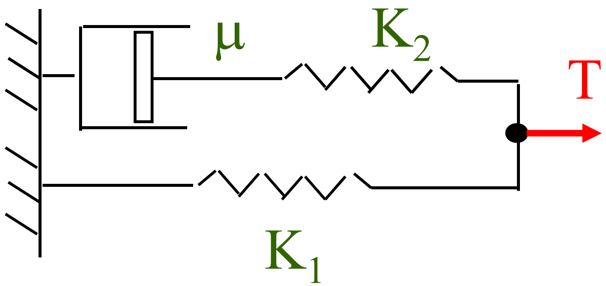
$$d = d_1 = d_2$$

Creep function :

d(t), for fixed T (=T_0)

$$d = \frac{T_0}{K}[1 - \exp(-\frac{K}{\mu}t)]$$

Kelvin element :



$$T = T_1 + T_2$$

$$T_1 = K_1 d$$

$$\dot{d} = \frac{T_2}{\mu} + \frac{\dot{T}_2}{K_2}$$

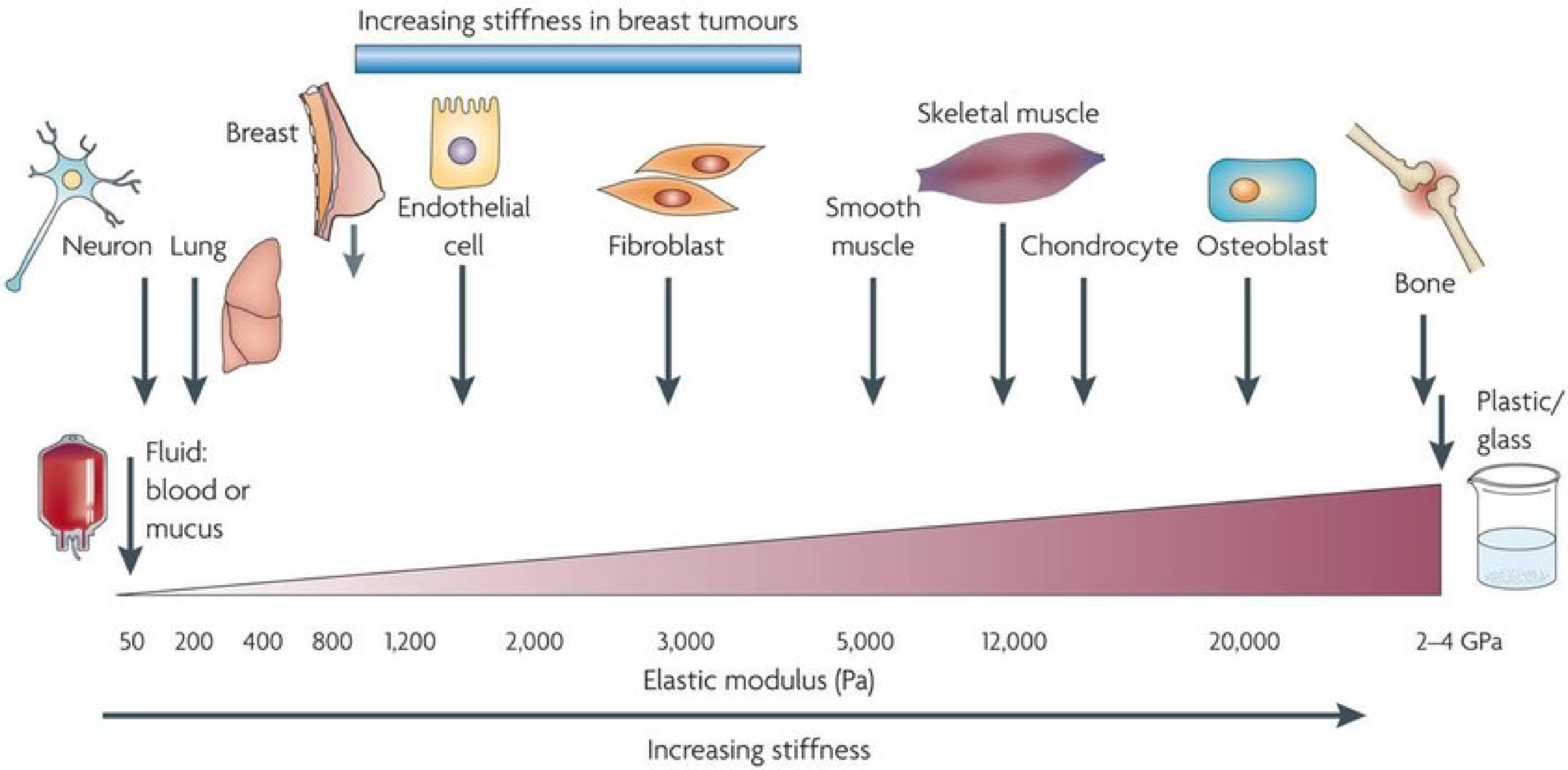
→

Stress-strain constitutive equation obtained from:

$$T + \frac{\mu}{K_2}\dot{T} = K_1 d + \mu (1 + \frac{K_1}{K_2})\dot{d}$$

Introduction to solid and soft tissues mechanics

Orders of magnitude

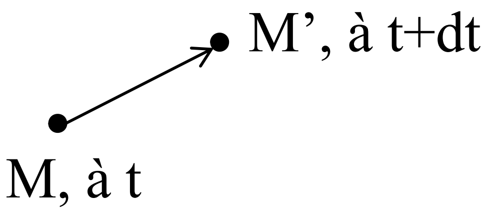


Strain tensor

Réponse à une contrainte:

pour solide:
une déformation

pour fluide:
une vitesse de déformation



Un point M du **solide** va être déplacé en M’; un point voisin N va être déplacé en N’

$\overrightarrow{MM'}$ est le vecteur déplacement en M; si il y a déformation du solide, le vecteur déplacement en M n’est pas le même que le vecteur déplacement en N. Le vecteur déplacement n’est pas le même aux différents endroits du solide : il dépend des coordonnées d’espace (x, y, z) (notées aussi (x_1, x_2, x_3)).

- Si on note:

$$\overrightarrow{MM'}\left|\begin{array}{l}\zeta_1\\ \zeta_2\\ \zeta_3\end{array}\right.$$

le tenseur des déformations (en petites déformations) est alors défini par:

$$\left[e_{ij}\right]=\frac{1}{2}\left(\frac{\partial \zeta_i}{\partial x_j}+\frac{\partial \zeta_j}{\partial x_i}\right)$$

Linear isotropic elastic solid: constitutive equation

Ecriture développée du tenseur des déformations:

$$[e_{ij}] = \begin{bmatrix} \dfrac{\partial \zeta_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial \zeta_1}{\partial x_2} + \dfrac{\partial \zeta_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \zeta_1}{\partial x_3} + \dfrac{\partial \zeta_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial \zeta_1}{\partial x_2} + \dfrac{\partial \zeta_2}{\partial x_1}\right) & \dfrac{\partial \zeta_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial \zeta_3}{\partial x_2} + \dfrac{\partial \zeta_2}{\partial x_3}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial \zeta_1}{\partial x_3} + \dfrac{\partial \zeta_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \zeta_3}{\partial x_2} + \dfrac{\partial \zeta_2}{\partial x_3}\right) & \dfrac{\partial \zeta_3}{\partial x_3} \end{bmatrix}$$

→ Trace du tenseur des déformations (notée θ):

$$\theta = \frac{\partial \zeta_1}{\partial x_1} + \frac{\partial \zeta_2}{\partial x_2} + \frac{\partial \zeta_3}{\partial x_3}$$

définit la dilatation volumique relative d'un parallélépipède de volume initial $\mathcal{V}_0$: $\mathcal{V} = (1+\theta)\mathcal{V}_0$

→ Loi de comportement du solide élastique linéaire:

$$\sigma_{ij} = \lambda\,\theta\,\delta_{ij} + 2\,G\,e_{ij}$$

λ et G sont des constantes caractéristiques du milieu considéré , dites constantes de Lamé (homogène à des contraintes (Pa))

ou bien : $$e_{ij} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})\,\delta_{ij} + \frac{(1+\nu)}{E}\sigma_{ij}$$

Mechanical parameters

E = module d'Young du matériau (en Pa)

ν = coefficient de Poisson G est dit aussi: module de cisaillement

- Relations entre ces grandeurs:

$$G = \frac{E}{2\,(1+\nu)} \qquad \lambda = \frac{\nu\, E}{(1+\nu)\,(1-2\nu)} \qquad \nu = \frac{\lambda}{2\,(\lambda+G)} \qquad E = \frac{G\,(3\lambda+2G)}{(\lambda+G)}$$

→ Interprétation physique simple du coeff. de Poisson:

Traction uni-axiale d'un cylindre de longueur initiale l_0 et rayon r_0 ; l et r sont la longueur et le rayon après traction

Alors: $$\frac{r-r_0}{r_0} = -\nu\left(\frac{l-l_0}{l_0}\right) = -\nu\,\varepsilon$$

ν définit donc la variation de rayon qui résulte de la variation de longueur du cylindre

Le volume du cylindre est : $V = l\,\pi\, r^2$

En écriture différentielle, ceci donne: $$\frac{\Delta V}{V_0} = \frac{\Delta l}{l_0} + 2\frac{\Delta r}{r_0} = (1-2\nu)\,\varepsilon$$

Ainsi, pour un matériau incompressible, on aura : ΔV = 0 et ν = 0.5

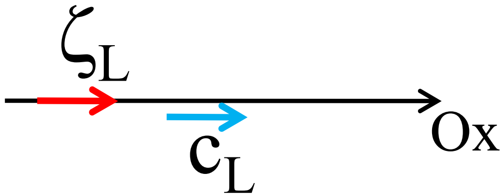
Elastic wave propagation in a continuous medium

→ Une onde engendre dans un milieu continu un déplacement transitoire MM’ de faible amplitude et de faible vitesse, mais qui se propage avec une célérité c élevée.

→ Une onde élastique qui se propage dans un milieu isotrope est en fait constituée de deux ondes qui se propagent à des célérités différentes : c_L, pour l’onde longitudinale et c_t, pour l’onde transversale

Onde longitudinale

(ou « de compression »)



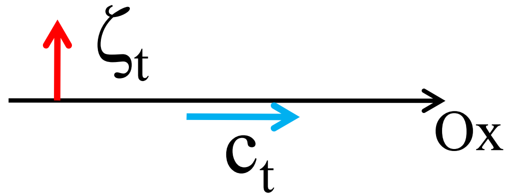
ζ_L = proj. longitudinale du vecteur **MM’**

$$\frac{\partial^2 \zeta_1}{\partial x^2} - \frac{\rho(1+\nu)(1-2\nu)}{E(1-\nu)} \frac{\partial^2 \zeta_1}{\partial t^2} = 0$$

$$= \frac{1}{c_L^2}$$

Onde transverse

(ou « de cisaillement »)



ζ_t = proj. transverse de **MM’**

$$\frac{\partial^2 \zeta_2}{\partial x^2} - \frac{2\rho(1+\nu)}{E} \frac{\partial^2 \zeta_2}{\partial t^2} = 0$$

$$\frac{\partial^2 \zeta_3}{\partial x^2} - \frac{2\rho(1+\nu)}{E} \frac{\partial^2 \zeta_3}{\partial t^2} = 0$$

Application pour la biomécanique des tissus mous (tumeurs, peau, foie, ligaments, cartilage, tendons, sein, graisse, …)

→ La célérité des ondes s'exprime en fonction des propriétés mécaniques du matériau:

$$c_L^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)} \quad \text{et} \quad c_t^2 = \frac{E}{2\rho(1+\nu)}$$

Donc, si on mesure les célérités, on peut en déduire E et ν

→ Principe de l'**élastographie ultra-sonore**
(propagation des ultra-sons dans le corps)